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MATHEMATICS
TARGET IIT JEE

THREE DIMENSIONAL GEOMETRY (3-D)

THEORY AND EXERCISE BOOKLET

CONTENTS

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JEE Syllabus :

Direction cosines and direction ratios, equation of a straight line in space, equation of a plane, distance of a point from a plane.

A. DISTANCE BETWEEN TWO POINTS

Let P and Q be two given points in space. Let the co-ordinates of the points P and Q be (x_1, y_1 z_1) and (x_2, y_2, z_2) with respect to a set OX, OY, OZ of rectangular axes. The position vectors of the points P and Q are given by $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Now we have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. $= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$$= (x_2 - x_1)\,\hat{i} - (y_2 - y_1)\,\hat{j} - (z_2 - z_1)\,\hat{k}.$$

$\therefore$ $PQ = |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

B. SECTION FORMULA

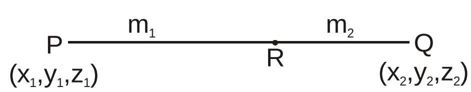
$$x = \frac{m_2x_1 + m_1x_2}{m_1 + m_2} \quad ; \quad y = \frac{m_2y_1 + m_1y_2}{m_1 + m_2} \quad ; \quad z = \frac{m_2z_1 + m_1z_2}{m_1 + m_2}$$

(for external division take –ve sign)

To determine the co-ordinates of a point R which divides the joining of two points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) internally in the ratio $m_1 : m_2$. Let OX, OY, OZ be a set of rectangular axes.

The position vectors of the two given points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are given by

$\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ ....(1) and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ ....(2)



Also if the co-ordinates of the point R are (x, y, z), then $\overrightarrow{OR} = x\hat{i} + y\hat{j} + z\hat{k}$. .....(3)

Now the point R divides the join of P and Q in the ratio $m_1 : m_2$, so that

Hence $m_2\overrightarrow{PR} = m_1\overrightarrow{RQ}$ or $m_2(\overrightarrow{OR} - \overrightarrow{OP}) = m_1(\overrightarrow{OQ} - \overrightarrow{OR})$ or $\overrightarrow{OR} = \dfrac{m_1\overrightarrow{OQ} + m_2\overrightarrow{OP}}{m_1 + m_2}$

or $x\hat{i} + y\hat{j} + z\hat{k} = \dfrac{(m_1x_2 + m_2x_1)\hat{i} + (m_1y_2 + m_2y_1)\hat{j} + (m_1z_2 + m_2z_1)\hat{k}}{(m_1 + m_2)}$ [Using (1), (2) and (3)]

Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ we get $x = \dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}$, $y = \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}$, $z = \dfrac{m_1z_2 + m_2z_1}{m_1 + m_2}$

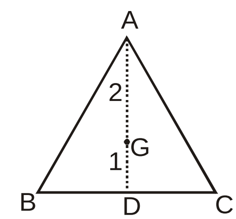
Remark : The middle point of the segment PQ is obtained by putting $m_1 = m_2$. Hence the co-ordinates of the middle point of PQ are $\left(\dfrac{1}{2}(x_1 + x_2), \dfrac{1}{2}(y_1 + y_2), \dfrac{1}{2}(z_1 + z_2)\right)$

CENTROID OF A TRIANGLE :

Let ABC be a triangle. Let the co-ordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Let AD be a median of the ΔABC. Thus D is the mid point of BC.

$\therefore$ The co-ordinates of D are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right)$

Now if G is the centroid of ΔABC, then G divides AD in the ratio 2 : 1. Let the co-ordinates of G be (x, y, z). Then $x = \frac{2.\left(\frac{x_2 + x_3}{2}\right) + 1.x_1}{2 + 1}$, or $x = \frac{x_1 + x_2 + x_3}{3}$.



Similarly $y = \frac{1}{2}(y_1 + y_2 + y_3)$, $z = \frac{1}{2}(z_1 + z_2 + z_3)$.

CENTROID OF A TETRAHEDRON :

Let ABCD be a tetrahedron, the co-ordinates of whose vertices are (x_r, y_r, z_r), r = 1, 2, 3, 4.
Let G_1 be the centroid of the face ABC of the tetrahedron. Then the co-ordinates of G_1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

The fourth vertex D of the tetrahedron does not lie in the plane of ΔABC. We know from statics that the centroid of the tetrahedron divides the line DG_1 in the ratio 3 : 1. Let G be the centroid of the tetrahedron and if (x, y, z) are its co-ordinates, then

$$x = \frac{3.\frac{x_1 + x_2 + x_3}{3} + 1.x_4}{3 + 1} \text{ or } x = \frac{x_1 + x_2 + x_3 + x_4}{4}. \text{ Similarly } y = \frac{1}{4}(y_1 + y_2 + y_3 + y_4),\ z = \frac{1}{4}(z_1 + z_2 + z_3 + z_4).$$

Ex.1 P is a variable point and the co-ordinates of two points A and B are (–2, 2, 3) and (13, –3, 13) respectively. Find the locus of P if 3PA = 2PB.

Sol. Let the co-ordinates of P be (x, y, z).

$\therefore$ PA = $\sqrt{(x+2)^2 + (y-2)^2 + (z-3)^2}$ ....(1) and PB = $\sqrt{(x-13)^2 + (y+3)^2 + (z-13)^2}$ ....(2)

Now it is given that 3PA = 2PB i.e., $9PA^2 = 4PB^2$. ....(3)

Putting the values of PA and PB from (1) and (2) in (3), we get

$9\{(x + 2)^2 + (y - 2)^2 + (z - 3)^2\} = 4\{(x - 13)^2 + (y + 3)^2 + (z - 13)^2\}$

or $9\{x^2 + y^2 + z^2 + 4x - 4y - 6z + 17\} = 4\{x^2 + y^2 + z^2 - 26x + 6y - 26z + 347\}$

or $5x^2 + 5y^2 + 5z^2 + 140x - 60y + 50z - 1235 = 0$ or $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

This is the required locus of P.

Ex.2 Find the ratio in which the xy-plane divides the join of (–3, 4, –8) and (5, –6, 4). Also find the point of intersection of the line with the plane.

Sol. Let the xy-plane (i.e., z = 0 plane) divide the line joining the points (–3,4, –8) and (5, –6, 4) in the ratio μ : 1, in the point R. Therefore, the co-ordinates of the point R are

$$\left(\frac{5\mu-3}{\mu+1}, \frac{-6\mu+4}{\mu+1}, \frac{4\mu-8}{\mu+1}\right) \quad ....(1)$$

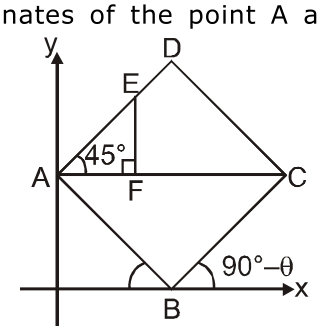
But on xy-plane, the z co-ordinate of R is zero

$\therefore$ $(4\mu - 8) / (\mu + 1) = 0$, or $\mu = 2$. Hence $\mu : 1 = 2 : 1$. Thus the required ratio is 2 : 1.

Again putting $\mu = 2$ in (1), the co-ordinates of the point R become (7/3, –8/3, 0).

Ex.3 ABCD is a square of side length 'a'. Its side AB slides between x and y-axes in first quadrant. Find the locus of the foot of perpendicular dropped from the point E on the diagonal AC, where E is the midpoint of the side AD.

Sol. Let vertex A slides on y-axis and vertex B slides on x-axis coordinates of the point A are $(0, a \sin\theta)$ and that of C are $(a \cos\theta + a \sin\theta, a \cos\theta)$



In ΔAEF, AF = $\frac{a}{2}\cos 45° = \frac{a}{2\sqrt{2}}$ and FC = AC – AF = $\sqrt{2}a - \frac{a}{2\sqrt{2}} = \frac{3a}{2\sqrt{2}}$

$\Rightarrow$ AF : FC = $\frac{a}{2\sqrt{2}} : \frac{3a}{2\sqrt{2}}$ = 1 : 3

$\Rightarrow$ Let the coordinates of the point F are (x, y)

$$\Rightarrow \quad x = \frac{3\times 0 + 1(a\cos\theta + a\sin\theta)}{4} = \frac{a(\sin\theta + \cos\theta)}{4}$$

$\Rightarrow$ $\frac{4x}{a} = \sin\theta + \cos\theta$ ....(1) and $y = \frac{3a\sin\theta + a\cos\theta}{4}$ $\Rightarrow$ $\frac{4y}{a} = 3\sin\theta + \cos\theta$...(2)

Form (1) and (2), $\sin\theta = \frac{2(y-x)}{a}$ and $\cos\theta = \frac{6x-2y}{a}$

$\Rightarrow$ $(y - x)^2 + (3x - y)^2 = \frac{a^2}{4}$ is the locus of the point F.

C. DIRECTION COSINES OF A LINE

If α, β, γ are the angles which a given directed line makes with the positive directions of the axes. of x, y and z respectively, then cos α, cos β cos γ are called the direction cosines (briefly written as d.c.'s) of the line. These d.c.'s are usually denote by ℓ, m, n.

Let AB be a given line. Draw a line OP parallel to the line AB and passing through the origin O. Measure angles α, β, γ, then cos α, cos β, cos γ are the d.c.'s of the line AB. It can be easily seen that ℓ, m, n, are the direction cosines of a line if and only if $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector in the direction of that line.

Clearly OP′(i.e. the line through O and parallel to BA) makes angle 180° – α, 180°– β, 180° – γ with OX, OY and OZ respectively. Hence d.c.'s of the line BA are cos (180° – α), cos (180° – β), cos (180° – γ) i.e., are –cos α, –cos β, – cos γ.

If the length of a line OP through the origin O be r, then the co-ordinates of P are (ℓr, mr, nr) where ℓ, m, n are the d c.'s of OP.

If ℓ, m, n are direction cosines of any line AB, then they will satisfy $\ell^2 + m^2 + n^2 = 1$.

DIRECTION RATIOS :

If the direction cosines ℓ, m, n of a given line be proportional to any three numbers a, b, c respectively, then the numbers a, b, c are called direction ratios (briefly written as d.r.'s of the given line.

RELATION BETWEEN DIRECTION COSINES AND DIRECTION RATIOS :

Let a, b, c be the direction ratios of a line whose d.c.'s are ℓ, m, n. From the definition of d.r.'s. we have $\ell/a = m/b = n/c = k$ (say). Then $\ell = ka$, $m = kb$, $n = kc$. But $\ell^2 + m^2 + n^2 = 1$.

$\therefore$ $k^2 (a^2 + b^2 + c^2) = 1$, or $k^2 = 1/(a^2 + b^2 + c^2)$ or $k = \pm \dfrac{1}{\sqrt{(a^2+b^2+c^2)}}$.

Taking the positive value of k, we get $\ell = \dfrac{a}{\sqrt{(a^2+b^2+c^2)}}$, $m = \dfrac{b}{\sqrt{(a^2+b^2+c^2)}}$, $n = \dfrac{c}{\sqrt{(a^2+b^2+c^2)}}$

Again taking the negative value of k, we get $\ell = \dfrac{-a}{\sqrt{(a^2+b^2+c^2)}}$, $m = \dfrac{-b}{\sqrt{(a^2+b^2+c^2)}}$, $n = \dfrac{-c}{\sqrt{(a^2+b^2+c^2)}}$.

Remark. Direction cosines of a line are unique. But the direction ratios of a line are by no means unique. If a, b, c are direction ratios of a line, then ka, kb, kc are also direction ratios of that line where k is any non-zero real number. Moreover if a, b, c are direction ratios of a line, then $a\hat{i} + b\hat{j} + c\hat{k}$ is a vector parallel to that line.

Ex.4 Find the direction cosines $\ell + m + n$ of the two lines which are connected by the relation $\ell + m + n = 0$ and $mn - 2n\ell - 2\ell m = 0$.

Sol. The given relations are $\ell + m + n = 0$ or $\ell = -m - n$ ....(1) and $mn - 2n\ell - 2\ell m = 0$ ...(2)

Putting the value of ℓ from (1) in the relation (2), we get

$mn - 2n(-m - n) - 2(-m - n)m = 0$ or $2m^2 + 5mn + 2n^2 = 0$ or $(2m + n)(m + 2n) = 0$.

$\therefore$ $\dfrac{m}{n} = -\dfrac{1}{2}$ and -2. From (1), we have $\dfrac{\ell}{n} = \dfrac{-m-n}{n} = -\dfrac{m}{n} - 1$ ...(3)

Now when $\dfrac{m}{n} = -\dfrac{1}{2}$, (3) given $\dfrac{\ell}{n} = \dfrac{1}{2} - 1 = -\dfrac{1}{2}$. $\therefore$ $\dfrac{m}{1} = \dfrac{n}{-2}$ and $\dfrac{\ell}{1} = \dfrac{n}{-2}$

i.e. $\dfrac{\ell}{1} = \dfrac{m}{1} = \dfrac{n}{-2} = \dfrac{\sqrt{(\ell^2+m^2+n^2)}}{\sqrt{\{1^2+1^2+(-2)^2\}}} = \dfrac{1}{\sqrt{6}}$ $\therefore$ The d.c.'s of one line are $\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}$.

Again when $\dfrac{m}{2} = -2$, (3) given $\dfrac{\ell}{n} = 2 - 1 = 1$.

i.e. $\dfrac{\ell}{1} = \dfrac{m}{-2} = \dfrac{n}{1} = \dfrac{\sqrt{(\ell^2+m^2+n^2)}}{\sqrt{\{1^2+(-2)^2+1^2\}}} = \dfrac{1}{\sqrt{6}}$ $\therefore$ The d.c.'s of the other line are $\dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}$.

To find the projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the another line whose d.c.'s are ℓ, m, n.

Let O be the origin. Then $\overrightarrow{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

$\therefore\ \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

Now the unit vector along the line whose d.c.'s are ℓ,m,n $= \ell\hat{i} + m\hat{j} + n\hat{k}$.

$\therefore$ projection of PQ on the line whose d.c.'s are ℓ, m, n

$= [(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}] \cdot (\ell\hat{i} + m\hat{j} + n\hat{k}) = \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

The angle θ between these two lines is given by $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$

If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are two sets of real numbers, then

$(\ell_1^2 + m_1^2 + n_1^2)(\ell_2^2 + m_2^2 + n_2^2) - (\ell_1\ell_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - n_2\ell_1)^2 + (\ell_1m_2 - \ell_2m_1)^2$

Now, we have

$\sin^2\theta = 1 - \cos^2\theta = 1 - (\ell_1\ell_2 + m_1m_2 + n_1n_2)^2 = (\ell_1^2 + m_1^2 + n_1^2)(\ell_2^2 + m_2^2 + n_2^2) - (\ell_1\ell_2 + m_1m_2 + n_1n_2)^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - n_2\ell_1)^2 + (\ell_1m_2 - \ell_2m_1)^2 = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & n_1 \\ \ell_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix}^2$

Condition for perpendicularity $\Rightarrow\ \ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

Condition for parallelism $\Rightarrow\ \ell_1 = \ell_2,\ m_1 = m_2,\ n_1 = n_2. \Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Ex.5 Show that the lines whose d.c.'s are given by $\ell + m + n = 0$ and $2mn + 3\ell n - 5\ell m = 0$ are at right angles.

Sol. From the first relation, we have $\ell = -m - n$. ....(1)

Putting this value of ℓ in the second relation, we have

$2mn + 3(-m - n)n - 5(-m - n)m = 0$ or $5m^2 + 4mn - 3n^2 = 0$ or $5(m/n)^2 + 4(m/n) - 3 = 0$ ....(2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the d,c,'s of the two lines. Then the roots of (2) are m_1/n_1 and m_2/n_2.

$\therefore$ product of the roots $= \dfrac{m_1}{n_1}\cdot\dfrac{m_2}{n_2} = -\dfrac{3}{5}$ or $\dfrac{m_1m_2}{3} = \dfrac{n_1n_2}{-5}$. ....(3)

Again from (1), $n = -\ell - m$ and putting this value of n in the second given relation, we have

$2m(-\ell - m) + 3\ell(-\ell - m) - 5\ell m = 0$ or $3(\ell/m)^2 + 10(\ell/m) + 2 = 0$.

$\therefore\ \dfrac{\ell_1}{m_1}\cdot\dfrac{\ell_2}{m_2} = \dfrac{2}{3}$ or $\dfrac{\ell_1\ell_2}{2} = \dfrac{m_1m_2}{3}$ From (3) and (4) we have $\dfrac{\ell_1\ell_2}{2} = \dfrac{m_1m_2}{3}\ \dfrac{n_1n_2}{-5} = k$ (say)

$\therefore\ \ell_1\ell_2 + m_1m_2 + n_1n_2 = (2 + 3 - 5)k = 0 \cdot k = 0. \Rightarrow$ The lines are at right angles.

Remarks :

(a) Any three numbers a, b, c proportional to the direction cosines are called the direction ratios

i.e. $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$ same sign either +ve or –ve should be taken throughout.

Note that d.r's of a line joining x_1, y_1, z_1 and x_2, y_2, z_2 are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$

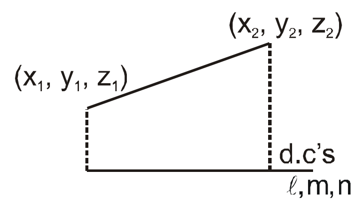
(b) If θ is the angle between the two lines whose d.c's are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2

$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

Hence if lines are perpendicular then $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$.

if lines are parallel then $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Note that if three lines are coplanar then $\begin{vmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{vmatrix} = 0$



(c) Projection of the join of two points on a line with d.c's ℓ, m, n are

$\ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

(d) If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are the d.c.'s of two concurrent lines, show that the d.c.'s of two lines bisecting the angles between them are proportional to $\ell_1 \pm \ell_2$, $m_1 \pm m_2$, $n_1 \pm n_2$.

D. AREA OF A TRIANGLE

Show that the area of a triangle whose vertices are the origin and the points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) is $\frac{1}{2}\sqrt{(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2}$.

The direction ratios of OA are x_1, y_1, z_1 and those of OB are x_2, y_2, z_2.

Also OA = $\sqrt{(x_1-0)^2 + (y_1-0)^2 + (z_1-0)^2} = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$

and OB = $\sqrt{(x_2-0)^2 + (y_2-0)^2 + (z_2-0)^2} = \sqrt{(x_2^2 + y_2^2 + z_2^2)}$.

$\therefore$ the d.c.' s of OA are $\frac{x_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}, \frac{y_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}, \frac{z_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}$

and the d.c.'s of OB are $\frac{x_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}, \frac{y_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}, \frac{z_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}$

Hence if θ is the angle between the line OA and OB, then

$$\sin\theta = \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}}{\sqrt{(x_1^2+y_1^2+z_1^2)}\sqrt{(x_2^2+y_2^2+z_2^2)}} = \frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_1)^2\}}}{OA.OB}$$

Hence the area of $\Delta OAB \;=\; \frac{1}{2}.\,OA\,.\,OB\sin\theta \qquad [\because \angle AOB = \theta]$

$$= \frac{1}{2}\,.\,OA.\,OB.\,\frac{\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}}{OA.OB} \qquad = \frac{1}{2}\sqrt{\{\Sigma(y_1z_2 - y_2z_2)^2\}}\,.$$

Ex.6 Find the area of the triangle whose vertices are A(1, 2, 3), B(2, –1, 1)and C(1, 2, –4).

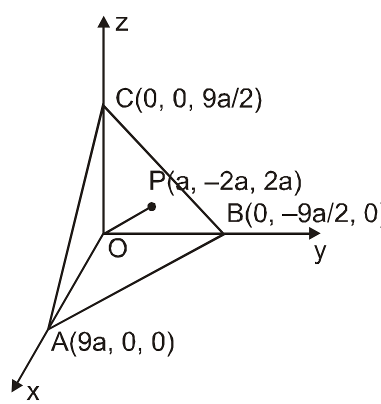
Sol. Let Δ_x, Δ_y, Δ_z be the areas of the projections of the area Δ of triangle ABC on the yz, zx and xy-planes respectively. We have

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & -4 & 1 \end{vmatrix} = \frac{21}{2} \quad ; \quad \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & -4 & 1 \end{vmatrix} = \frac{7}{2}$$

$$\Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \quad \therefore \text{ the required area } \Delta = \sqrt{[\Delta_x^2 + \Delta_y^2 + \Delta_z^2]} = \frac{7\sqrt{10}}{2} \text{ sq. units.}$$

Ex.7 A plane is passing through a point P(a, –2a, 2a), $a \neq 0$, at right angle to OP, where O is the origin to meet the axes in A, B and C. Find the area of the triangle ABC.

Sol. $OP = \sqrt{a^2 + 4a^2 + 4a^2} = |3a|$.



Equation of plane passing through P(a, –2a, 2a) is
$A(x - a) + B(y + 2a) + C(z - 2a) = 0$.
$\because$ the direction cosines of the normal OP to the plane ABC are proportional to
$a - 0,\ -2a - 0,\ 2a - 0$ i.e. $a, -2a, 2a$.
$\Rightarrow$ equation of plane ABC is
$a(x - a) - 2a(y + 2a) + 2a(z - 2a) = 0$
or $ax - 2ay + 2az = 9a^2$ ....(1)
Now projection of area of triangle ABC on ZX, XY and YZ planes are the triangles AOC, AOB and BOC respectively.
$\therefore (\text{Area } \Delta ABC)^2 = (\text{Area } \Delta AOC)^2 + (\text{Area } \Delta AOB)^2 + (\text{Area } \Delta BOC)^2$

$$= \left(\frac{1}{2}.\,AO\,.\,OC\right)^2 + \left(\frac{1}{2}.\,AO\,.\,BO\right)^2 + \left(\frac{1}{2}.\,BO\,.\,OC\right)^2$$

$$= \frac{1}{4}\left[\left(9a.\frac{9}{2}a\right)^2 + \left(9a\,.\frac{-9}{2}a\right)^2 + \left(\frac{-9}{2}a.\frac{9}{2}a\right)^2\right] = \frac{1}{4},\frac{81^2a^4}{4}\left(1 + 1 + \frac{1}{4}\right)$$

$$\Rightarrow \quad (\text{Area } \Delta ABC)^2 = \frac{9^5}{4^3}a^4 \Rightarrow \text{Area of } \Delta ABC = \frac{3^5}{2^3}a^2 = \frac{243}{8}a^2\,.$$

E. PLANE

(i) General equation of degree one in x, y, z i.e. $ax + by + cz + d = 0$ represents a plane.

(ii) Equation of a plane passing through (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

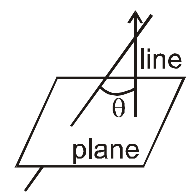
(iii) Equation of a plane if its intercepts on the co-ordinate axes are x_1, y_1, z_1 is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$.

(iv) Equation of a plane if the length of the perpendicular from the origin on the plane is 'p' and d.c's of the perpendiculars as ℓ, m, n is $\ell x + my + nz = p$

(v) **Parallel and perpendicular planes :**
Two planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are

Perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$, **parallel** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ and **Coincident** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$

(vi) Angle between a plane and a line is the complement of the angle between the normal to the plane and the line. If $\left.\begin{array}{ll} \text{Line} : & \vec{r} = \vec{a} + \lambda\vec{b} \\ \text{Plane} : & \vec{r}.\vec{n} = d \end{array}\right]$ then $\cos(90 - \theta) = \sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}$.



where θ is the angle between the line and normal to the plane.

(vii) Length of the $\perp^{ar}$ from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is $p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$

(viii) Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

(ix) Planes bisecting the angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2 x + b_2y + c_2 z + d_2 = 0$ is given by $\left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \pm\left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$ of these two bisecting planes, one bisects the acute and the other obtuse angle between the given planes.

(x) Equation of a plane through the intersection of two planes P_1 and P_2 is given by $P_1 + \lambda P_2 = 0$

Ex.8 Reduce the equation of the plane x + 2y − 2z − 9 = 0 to the normal form and hence find the length of the perpendicular drawn form the origin to the given plane.

Sol. The equation of the given plane is x + 2y − 2z − 9 = 0

Bringing the constant term to the R.H.S., the equation becomes x + 2y − 2z = 9 ...(1)

[Note that in the equation (1) the constant term 9 is positive. If it were negative, we would have changed the sign throughout to make it positive.]

Now the square root of the sum of the squares of the coefficients of x, y, z in (1)

$= \sqrt{(1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$.

Dividing both sides of (1) by 3, we have $\frac{1}{2}x + \frac{2}{3}y - \frac{2}{3}z = 3$. ....(2)

The equation (2) of the plane is in the normal form $\ell x + my + nz = p$.

Hence the d.c.'s ℓ, m, n of the normal to the plane are $\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}$ and the length p of the perpendicular from the origin to the plane is 3.

Ex.9 Find the equation to the plane through the three points (0, −1, −1), (4, 5, 1) and (3, 9, 4).

Sol. The equation of any plane passing through the point (0, −1, −1) is given by

$a(x - 0) + b\{y - (-1)\} + c\{z - (-1)\} = 0$ or $ax + b(y + 1) + c(z + 1) = 0$ ....(1)

If the plane (1) passes through the point (4, 5, 1), we have 4a + 6b + 2c = 0 ....(2)

If the plane (1) passes through the point (3, 9, 4), we have 3a + 10b + 5c = 0 ....(3)

Now solving the equations (2) and (3), we have $\frac{a}{30-20} = \frac{b}{6-20} = \frac{c}{40-18} = \lambda$ (say).

$\therefore$ a = 10λ, b = −14λ, c = 22λ.

Putting these value of a, b, c in (1), the equation of the required plane is given by

λ[10x − 14(y + 1) + 22(z + 1)] = 0 or 10x − 14(y + 1) + 22(z + 1) = 0 or 5x − 7y + 11z + 4 = 0.

Ex.10 Find the equation of the plane through (1, 0, −2) and perpendicular to each of the planes

2x + y − z − 2 = 0 and x − y − z − 3 = 0.

Sol. The equation of any plane through the point (1, 0, −2) is

a (x − 1) + b (y − 0) + c(z + 2) = 0. ...(1)

If the plane (1) is perpendicular to the planes 2x + y − z − 2 = 0 and x − y − z − 3 = 0, we have

a (2) + b(1) + c(−1) = 0 i.e., 2a + b − c = 0, ...(2)

and a(1) + b(−1) + c(−1) = 0 i.e., a − b − c = 0. ...(3)

Adding the equation (2) and (3), we have $c = \frac{3}{2}a$. Subtracting (3) from (2), we have $b = -\frac{1}{2}a$.

Putting the values of b and c in (1), the equation of the required plane is given by

$a(x - 1) - \frac{1}{2}ay + \frac{3}{2}a(z + 2) = 0$ or 2x − 2 − y + 3z + 6 = 0 or 2x − y + 3z + 4 = 0.

Ex.11 Find the equation of the plane passing through the line of intersection of the planes $2x - 7y + 4z = 3$, $3x - 5y + 4z + 11 = 0$, and the point $(-2, 1, 3)$

Sol. The equation of any plane through the line of intersection of the given plane is

$$(2x - 7y + 4z - 3) + \lambda (3x - 5y + 4z + 11) = 0. \qquad ....(1)$$

If the plane (1) passes through the point $(-2, 1, 3)$, then substituting the co-ordinates of this point in the equation (1), we have

$$\{2(-2) - 7(1) + 4(3) - 3\} + \lambda\{3 (-2) - 5(1) + 4(3) + 1\} = 0 \text{ or } (-2) + \lambda(12) = 0 \text{ or } \lambda = 1/6.$$

Putting this value of λ in (1), the equation of the required plane is

$(2x - 7y + 4z - 3) + (1/6) (3x - 5y + 4z + 11) = 0$ or $15x - 47y + 28z = 7$.

Ex.12 A variable plane is at a constant distance 3p from the origin and meets the axes in A, B and C. Prove that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Sol. Let the equation of the variable plane be $x/a + y/b + z/c = 1$. .....(1)

It is given that the length of the perpendicular from the origin to the plane (1) is 3p.

$$\therefore \quad 3p = \frac{1}{\sqrt{(1/a^2 + 1/b^2 + 1/c^2)}} \text{ or } \frac{1}{9p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}, \qquad .....(2)$$

The plane (1) meets the coordinate axes in the points A, B and C whose co-ordinates are respectively given by (a, 0, 0), (0, b, 0) and (0, 0, c). Let (x, y, z) be the co-ordinates of the centroid of the triangle ABC. Then $x = (a + 0 + 0)/3$, $y = (0 + b + 0)/3$, $z = (0 + 0 + c)/3$

i.e., $x = \frac{1}{3}a$, $y = \frac{1}{3}b$, $z = \frac{1}{3}c$. $\therefore$ $a = 3x, b = 3y, c = 3z$. .....(3)

The locus of the centroid of the triangle ABC is obtained by eliminating a, b, c between the equation (2) and (3). Putting the value of a, b, c from (3) in (2), the required locus is given by

$$\frac{1}{9p^2} = \frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} \text{ or } x^{-2} + y^{-2} + z^{-2} = p^{-2}.$$

Ex.13 Show that the origin lies in the acute angles between the planes $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$. Find the planes bisecting the angles between them and point out the one which bisects the acute angle.

Sol. In order that the constant terms are positive, the equations of the given planes may be written as

$$-x - 2y - 2z + 9 = 0 \quad ...(1) \qquad \text{and} \quad 4x - 3y + 12z + 13 = 0.$$

We have $a_1a_2 + b_1b_2 + c_1c_2 = (-1).4 + (-2). (-3) + (-2).(12) = - 4 + 6 - 24 = - 22$ = negative.

Hence the origin lies in the acute angle between the planes (1) and (2)

The equation of the plane bisecting the angle between the given planes (1) and (2) when contains the origin is $\frac{x - 2y - 2z + 9}{\sqrt{(1 + 4 + 4)}} = \frac{4x - 3y + 12z + 13}{\sqrt{(16 + 9 + 144)}}$

or $13 (-x - 2y - 2z + 9) = 3(4x - 3y + 12z + 13)$ or $25x + 17y + 62z - 78 = 0$ ...(3)

We have proved above that origin lies in the acute angle between the planes and so the equation (3) is the equation of the bisector plane which bisects the acute angle between the given planes.

The equation of the other bisector plane (i.e., the plane bisecting the obtuse angle) is

$$-\frac{x-2y-2z+9}{\sqrt{(1+4+4)}} = -\frac{4x-3y+12z+13}{\sqrt{(16+9+144)}} \quad \text{or} \quad x + 35y - 10z - 156 = 0 \qquad ....(4)$$

the equation (3) and (4) given the planes bisecting the angle between the given planes and the equation (3) is the bisector of the acute angle.

Ex.14 The mirror image of the point (a, b, c) about coordinate planes xy, xz and yz are A, B and C. Find the orthocentre of the triangle ABC.

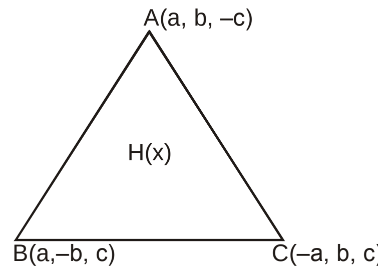
Sol. Let the point P be (a, b, c) $\Rightarrow$ A $\equiv$ (a, b, –c), B $\equiv$ (a, –b, c) and C $\equiv$ (–a, b, c)

Let the orthocentre of ΔABC be H $\equiv$ (x, y, z)

$\Rightarrow$ $(x - a)(2a) + (y - b)(-2b) + (z + c)\, 0 = 0$ $\Rightarrow$ $ax - by = a^2 - b^2$ ...(1)

Similarly, $by - cz = b^2 - c^2$ ...(2)

Also $\begin{vmatrix} x-a & y-b & z+c \\ 0 & 2b & -2c \\ -2a & 0 & 2c \end{vmatrix} = 0$ (As A, B, C and H are coplanar)



$\Rightarrow$ $bcx + acy + abz = abc$ ...(3)

for solving (1), (2) and (3),

$$D = \begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ bc & ac & ab \end{vmatrix} = a^2b^2 + b^2c^2 + a^2c^2, \quad D_1 = \begin{vmatrix} a^2-b^2 & -b & 0 \\ b^2-c^2 & b & -c \\ abc & ac & ab \end{vmatrix} = a^2(b^2 + c^2) - b^2c^2$$

$\Rightarrow$ Similarly $D_2 = b^2(c^2 + a^2) - a^2c^2$ and $D_3 = c^2(a^2 + b^2) - a^2b^2$

$\Rightarrow$ Orthocentre is $H \equiv \left(\frac{a^2(b^2+c^2)-b^2c^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{b^2(c^2+a^2)-a^2c^2}{a^2b^2+b^2c^2+c^2a^2}, \frac{c^2(a^2+b^2)-a^2b^2}{a^2b^2+b^2c^2+c^2a^2}\right)$.

F. STRAIGHT LINE

(i) Equation of a line through $A(x_1, y_1, z_1)$ and having direction cosines ℓ, m, n are

$$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$ and the lines through (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

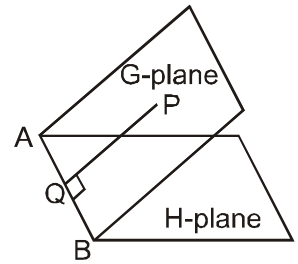
(ii) Intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.

(iii) General equation of the plane containing the line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is

$A(x - x_1) + B(y - y_1) + c(z - z_1) = 0$ where $A\ell + bm + cn = 0$.

(iv) Line of Greatest Slope

AB is the line of intersection of G-plane and H is the h orizontal plane. Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point 'P' perpendicular to the line of intersection of the given plane with any horizontal plane.



Ex.15 Show that the distance of the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$ is 13.

Sol. The equation of the given line are $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ = r (say). ....(1)

The co-ordinates of any point on the line (1) are $(3r + 2, 4r - 1, 12r + 2)$. If this point lies on the plane $x - y + z = 5$, we have $3r + 2 - (4r - 1) + 12r + 2 = 5$, or $11r = 0$, or $r = 0$.

Putting this value of r, the co-ordinates of the point of intersection of the line (1) and the given plane are $(2, -1, 2)$.

$\therefore$ The required distance = distance between the points $(2, -1, 2)$ and $(-1, -5, -10)$

$$= \sqrt{(2+1)^2+(-1+5)^2+(2+10)^2} = \sqrt{(9+16+144)}=\sqrt{(169)}=13$$

Ex.16 Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $3x + 4y - 6z + 1 = 0$. Find also the co-ordinates of the point on the line which is at the same distance from the foot of the perpendicular as the origin is.

Sol. The equation of the plane is $3x + 4y - 6z + 1 = 0$. ....(1)

The direction ratios of the normal to the plane (1) are 3, 4, −6. Hence the line normal to the plane (1) has d.r.'s 3, 4, −6, so that the equations of the line through (0, 0, 0) and perpendicular to the plane (1) are $x/3 = y/4 = z/-6 = r$ (say) ....(2)

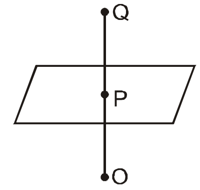
The co-ordinates of any point P on (2) are $(3r, 4r, -6r)$ ....(3)

If this point lies on the plane (1), then $3(3r) + r(4r) - 6(-6r) + 1 = 0$, or $r = -1/61$.

Putting the value of r in (3), the co-ordinates of the foot of the perpendicular P are $(-3/61, -4/61, 6/61)$.

Now let Q be the point on the line which is at the same distance from the foot of the perpendicular as the origin. Let (x_1, y_1, z_1) be the co-ordinates of the point Q. Clearly P is the middle point of OQ.

Hence we have $\frac{x_1+0}{2}=-\frac{3}{61},\frac{y_1+0}{2}=\frac{4}{61},\frac{z_1+0}{2}=\frac{6}{61}$



or $x_1 = 6/61, y_1 = -8/61, z_1 = 12/61$.

$\therefore$ The co-ordinates of Q are $(-6/61, -8/61, 12/61)$.

Ex.17 Find in symmetrical form the equations of the line 3x + 2y – z – 4 = 0 & 4x + y – 2z + 3 = 0 and find its direction cosines.

Sol. The equations of the given line in general form are 3x + 2y – z – 4 = 0 & 4x + y – 2z + 3 = 0 ..(1)

Let ℓ, m, n be the d.c.'s of the line. Since the line is common to both the planes, it is perpendicular to the normals to both the planes. Hence we have $3\ell + 2m - n = 0$, $4\ell + m - 2n = 0$.

Solving these, we get $\frac{\ell}{-4+1} = \frac{m}{-4+6} = \frac{n}{3-8}$ or $\frac{\ell}{-3} = \frac{m}{2} = \frac{n}{-5} = \frac{\sqrt{(\ell^2+m^2+n^2)}}{\sqrt{(9+4+25)}} = \frac{1}{\sqrt{38)}}$

$\therefore$ the d.c.'s of the line are $-\frac{3}{\sqrt{(38)}}, \frac{2}{\sqrt{(38)}}, -\frac{5}{\sqrt{(38)}}$.

Now to find the co-ordinates of a point on the line given by (1), let us find the point where it meets the plane z = 0. Putting z = 0 i the equations given by (1), we have 3x + 2y – 4 = 0, 4x + y + 3 = 0.

Solving these, we get $\frac{x}{6+4} = \frac{y}{-16-9} = \frac{1}{3-8}$, or x = –2, y = 5.

Therefore the equation of the given line in symmetrical form is $\frac{x+2}{-3} = \frac{y-5}{2} = \frac{z-0}{-5}$.

Ex.18 Find the equation of the plane through the line 3x – 4y + 5z = 10, 2x + 2y – 3z = 4 and parallel to the line x = 2y = 3z.

Sol. The equation of the given line are 3x – 4y + 5z = 10, 2x + 2y – 3z = 4 ...(1)

The equation of any plane through the line (1) is (3x – 4y + 5z – 10) + λ (2x + 2y – 3z – 4) = 0

or (3 + 2λ)x + (–4 +2λ) y + (5 – 3λ) z – 10 – 4λ = 0. ...(2)

The plane (1) will be parallel to the line x = 2y = 3z i.e. $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$ if

(3 + 2λ) . 6 + (–4 + 2λ). 3 + (5 – 3λ).2 = 0 or λ(12 + 6 – 6) + 18 – 12 + 10 = 0 or $\lambda = -\frac{4}{3}$.

Putting this value of λ in (2), the required equation of the plane is given by

$$\left(3-\frac{8}{3}\right)x + \left(-4-\frac{8}{3}\right)y + (5+4)z - 10 + \frac{16}{3} = 0 \quad \text{or} \quad x - 20y + 27z = 14.$$

Ex.19 Find the equation of a plane passing through the line $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-2}{-2}$ and making an angle of 30° with the plane x + y + z = 5.

Sol. The equation of the required plane is (x – y + 1) + λ (2y + z – 6) = 0 ⇒ x + (2λ – 1) y + λz + 1 – 6λ = 0

Since it makes an angle of 30° with x +y + z = 5

$$\Rightarrow \frac{|1+(2\lambda-1)+\lambda|}{\sqrt{3}.\sqrt{1+\lambda^2+(2\lambda-1)^2}} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad |6\lambda| = 3\sqrt{5\lambda^2-4\lambda+2} \quad \Rightarrow \quad 4\lambda^2 = 5\lambda^2 - 4\lambda + 2$$

$\Rightarrow \lambda^{2-} - 4\lambda + 2 = 0 \Rightarrow \lambda = (2 \pm \sqrt{2}) \Rightarrow (x - y + 1) + (2 \pm \sqrt{2})(2y + x - 6) = 0$ are two required planes.

Ex.20 Prove that the lines 3x + 2y + z – 5 = 0 = x + y – 2z – 3 and 2x – y – z = 0 = 7x + 10y – 8z – 15 are perpendicular.

Sol. Let ℓ_1, m_1, n_1 be the d.c.'s of the first line. Then $3\ell_1 + 2m_1 + n_1 = 0$, $\ell_1 + m_1 - 2n_1 = 0$. Solving, we get

$$\frac{\ell_1}{-4-1} = \frac{m_1}{1+6} = \frac{n_1}{3-2} \text{ or } \frac{\ell_1}{-5} = \frac{m_1}{7} = \frac{n_1}{1}.$$

Again let ℓ_2, m_2,n_2 be the d.c.'s of the second line, then $2\ell_2 - m_2 - n_2 = 0$, $7\ell_2 + 10m_2 - 8n_2 = 0$.

Solving, $\frac{\ell_2}{8+10} = \frac{m_2}{-7+16} = \frac{n_2}{20+7}$ or $\frac{\ell_2}{2} = \frac{m_2}{1} = \frac{n_2}{3}$.

Hence the d.c.'s of the two given lines are proportional to –5, 7, 1 and 2, 1, 3. We have

–5.2 + 7.1 + 1.3 = 0 $\therefore$ the given lines are perpendicular.

Ex.21 Find the equation of the plane which contains the two parallel lines

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1} \text{ and } \frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}.$$

Sol. The equation of the two parallel lines are

(x + 1)/3 = (y – 2)/2 = (z – 0)/1 ....(1) and (x – 3)/3 = (y + 4)/2 = (z – 1)/1. ....(2)

The equation of any plane through the line (1) is

a(x + 1) + b (y – 2) + cz = 0, ...(3) where 3a + 2b + c = 0. ....(4)

The line (2) will also lie on the plane (3) if the point (3, –4, 1) lying on the line (2) also lies on the plane (3), and for this we have a (3 + 1) + b (–4 – 2) + c. 1 = 0 or 4a – 6b + c = 0. ....(5)

Solving (4) and (5), we get $\frac{a}{8} = \frac{b}{1} = \frac{c}{-26}$.

Putting these proportionate values of a, b, c in (3), the required equation of the plane is

8(x + 1) + 1.(y – 2) – 26z = 0, or 8x + y – 26 + 6 = 0.

Ex.22 Find the distance of the point P(3, 8, 2) from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane 3x + 2y – 2z + 17 = 0.

Sol. The equation of the given line are (x – 1)/2 = (y – 3)/4 = (z – 2)/3 = r, (say). ...(1)

Any point Q on the line (1) is (2r + 1, 4r + 3, 3r + 2).

Now P is the point (3, 8, 2) and hence d.r.'s of PQ are

2r + 1 – 3, 4r + 3 – 8, 3r + 2 – 2 i.e. 2r – 2, 4r – 5, 3r.

It is required to find the distance PQ measured parallel to the plane 3x + 2y – 2z + 17 = 0 ...(2)

Now PQ is parallel to the plane (2) and hence PQ will be perpendicular to the normal to the plane (2).

Hence we have (2r – 2) (3) + (4r – 5) (2) + (2r) (–2) = 0 or 8r – 16 = 0, or r = 2.

Putting the value of r, the point Q is (5, 11, 8) = $\sqrt{[(3-5)^2 + (8-11)^2 + (2-8)^2]} = \sqrt{(4+9+36)} = 7$.

Ex.23 Find the projection of the line 3x – y + 2z = 1, x + 2y – z = 2 on the plane 3x + 2y + z = 0.

Sol. The equations of the given line are 3x – y + 2z = 1, x + 2y – z = 2. ....(1)

The equation of the given plane is 3x + 2y + z = 0. ....(2)

The equation of any plane through the line (1) is (3x – y + 2z – 1) + λ(x + 2y – z – 2) = 0

or (3 + λ) x + (–1 + 2λ) y + (2 – λ) z – 1 – 2λ = 0 ....(3)

The plane (3) will be perpendicular to the plane (2), if 3(3 + λ)+ 2(–1 + 2λ) + 1 (2 – λ) = 0 or $\lambda = -\frac{3}{3}$.

Putting this value of λ in (3), the equation of the plane through the line (1) and perpendicular to the plane (2) is given by $\left(3-\frac{3}{2}\right)$x + (–1 –3) y + $\left(2+\frac{3}{2}\right)$ z – 1 + 3 = 0 or 3x – 8y + 7z + 4 = 0. ....(4)

$\therefore$ The projection of the given line (1) on the given plane (2), is given by the equations (2) and (4) together.

Note : The symmetrical form of the projection given above by equations (2) and (4) is $\frac{x+\frac{4}{5}}{-11}=\frac{y-\frac{2}{5}}{9}=\frac{z}{5}$.

Ex.24 Find the image of the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ in the plane x + 2y + z = 12

Sol. Any point on the given line is 2r + 1, –r – 1, 4r + 3. If this point lies on the planes,

then 2r + 1 – 2r – 2 + 4r + 3 = 12 $\Rightarrow$ r = $\frac{5}{2}$.

Hence the point of intersection of the given line and that of the plane is $\left(6,-\frac{7}{2},13\right)$.

Also a point on the line is (1, –1, 3).

Let (α, β, γ) be its image in the given plane. In such a case $\frac{\alpha-1}{1}=\frac{\beta+1}{2}=\frac{\gamma-3}{1}=\lambda$

$\Rightarrow$ $\alpha = \lambda + 1$, $\beta = 2\lambda - 1$, $\gamma = \lambda + 3$. Now the midpoint of the image and the point (1, –1, 3) lies on the plane i.e. $\left(1+\frac{\lambda}{2},\lambda-1,3+\frac{\lambda}{2}\right)$ lies in the plane $\Rightarrow \lambda = \frac{10}{3}$. Hence the image of (1, –1, 3) is $\left(\frac{8}{3},\frac{7}{3},\frac{14}{3}\right)$.

Hence the equation of the required line is $\frac{x-6}{\frac{10}{3}}=\frac{y+\frac{7}{2}}{\frac{-35}{6}}=\frac{z-13}{\frac{25}{3}}$ or $\frac{x-6}{4}=\frac{y+\frac{7}{2}}{-7}=\frac{z-13}{10}$.

Ex.25 Find the foot and hence the length of the perpendicular from the point (5, 7, 3) to the line $(x - 15)/3 = (y - 29)/8 = (z - 5)/(-5)$. Find the equations of the perpendicular. Also find the equation of the plane in which the perpendicular and the given straight line lie.

Sol. Let the given point (5, 7, 3) be P.

The equations of the given line are $(x - 15)/3 = (y - 29)/8 = (z - 5)/(-5) = r$ (say). ...(1)

Let N be the foot of the perpendicular from the point P to the line (1). The co-ordinates of N may be taken as $(3r + 15, 8r + 29, -5r + 5)$. ...(2)

$\therefore$ the direction ratios of the perpendicular PN are

$3r + 15 - 5, 8r + 29 - 7, -5r + 5 - 3$, i.e. are $3r + 10, 8r + 22, -5r + 2$. ...(3)

Since the line (1) and the line PN are perpendicular to each other, therefore

$3(3r + 10) + 8(8r + 22) - 5(-5r + 2) = 0$ or $98r + 196 = 0$ or $r = -2$

Putting this value of r in (2) and (3), the foot of the perpendicular N is (9, 13, 15) and the direction ratios of the perpendicular PN are 4, 6, 12 or 2, 3, 6.

$\therefore$ the equations of the perpendicular PN are $(x - 5)/2 = (y - 7)/3 = (z - 3)/6$. ...(4)

Length of the perpendicular PN

= the distance between P(5, 7, 3) and N(9, 13, 15) = $\sqrt{(9-5)^2 + (13-7)^2 + (15-3)^2} = 14$.

Lastly the equation of the plane containing the given line (1) and the perpendicular (4) is given by

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

or $(x - 15)(48 + 15) - (y - 29)(18 + 10) + (z - 5)(9 - 16) = 0$ or $9x - 4y - z =- 14 = 0$.

Ex.26 Show that the planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$, $8x - 31y - 33z = 0$ pass through the one line find its equations.

Sol. The rectangular array of coefficient is $\begin{vmatrix} 2 & -3 & -7 & 0 \\ 3 & -14 & -13 & 0 \\ 8 & -31 & -33 & -0 \end{vmatrix}$.

We have, $\Delta_4 = \begin{vmatrix} 2 & -3 & -7 \\ 3 & -14 & -13 \\ 8 & -31 & -33 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -1 \\ 3 & -11 & -4 \\ 8 & -23 & -9 \end{vmatrix}$ (by $C_2 + C_1$, $C_2 + 3C_1$)

$= \begin{vmatrix} 0 & 0 & -1 \\ -5 & -7 & -4 \\ -10 & -14 & -9 \end{vmatrix} = -1(70 - 70) = 0,$ (by $C_1 + 2C_2$, $C_2 - C_2$)

since $\Delta_4 = 0$, therefore, the three planes either intersect in a line or form a triangular prism.

Now $\Delta_3 = \begin{vmatrix} 2 & -3 & 0 \\ 3 & -14 & 0 \\ 8 & -31 & 0 \end{vmatrix} = 0$ Similarly $\Delta_2 = 0$ and $\Delta_1 = 0$,

Hence the three planes intersect in a common line.

Clearly the three planes pass through (0, 0, 0) and hence the common line of intersection will pass through (0, 0, 0). The equations of the common line are given by any of the two given planes. Therefore the equations of the common line are given by $2x - 3y - 7z = 0$ and $3x - 14y - 13z = 0$.

$\therefore$ the symmetric form of the line is given by $\frac{x}{39-98} = \frac{y}{-21+26} = \frac{z}{-28+9}$ or $\frac{x}{-59} = \frac{y}{5} = \frac{z}{-19}$.

Ex.27 For what values of k do the planes $x - y + z + 1 = 0$, $kx + 3y + 2z - 3 = 0$, $3x + ky + z - 2 = 0$ **(i)** intersect in a point ; **(ii)** intersect in a line ; **(iii)** form a triangular prism ?

Sol. The rectangular array of coefficients is $\begin{vmatrix} 1 & -1 & 1 & 1 \\ k & 3 & 2 & 3 \\ 3 & k & 1 & -2 \end{vmatrix}$

Now we calculate the following determinants

$$\Delta_4 = \begin{vmatrix} 1 & -1 & 1 \\ k & 3 & 2 \\ 3 & k & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ k+3 & 3 & 5 \\ 3+k & k & k+1 \end{vmatrix} \qquad \text{(adding 2nd column to 1st and 3rd)}$$

$$= (k + 3) \begin{vmatrix} 0 & -1 & 0 \\ 1 & 3 & 5 \\ 1 & k & k+1 \end{vmatrix} = (k + 3) (k + 1 - 5) = (k + 3) (k - 4).$$

$$\Delta_2 = \begin{vmatrix} 1 & -1 & 1 \\ k & 3 & -3 \\ 3 & k & -2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ k+3 & 3 & 0 \\ 3+k & k & k-2 \end{vmatrix} = (k + 3) (k - 2), \text{ (adding 2nd column to 1st and 3rd)}$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ k & 2 & -3 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ k-2 & 2 & -5 \\ 2 & 1 & -3 \end{vmatrix} \qquad \text{(adding (−1) times 2nd column to 1st and 3rd)}$$

$$= -\{(k - 2) (-3) + 10\} = 3k - 16,$$

$$\text{and } \Delta_1 = \begin{vmatrix} -1 & 1 & 1 \\ 3 & 2 & -3 \\ k & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & -3 \\ k-2 & 1 & -2 \end{vmatrix} = -5 (k - 2) \qquad \text{(adding 3rd column to 1st)}$$

(i) The given planes will intersect in a point if $\Delta_4 \neq 0$ and so we must have $k \neq -3$ and $k \neq 4$. Thus the given planes will intersect in a point for all real values of k other than −3 and 4.

(ii) If $k = -3$, we have $\Delta_4 = 0$, $\Delta_3 = 0$ but $\Delta_2 \neq 0$. Hence the given planes will form a triangular prism if $k = -3$.

(iii) If $k = 4$, we have $\Delta_4 = 0$ but $\Delta_3 \neq 0$. Hence the given planes will form a triangular prism if $k = 4$. We observe that for no value of k the given planes will have a common line of intersection.

Ex.28 Find the equation of the line passing through (1, 1, 1) and perpendicular to the line of intersection of the planes $x + 2y - 4z = 0$ and $2x - y + 2z = 0$.

Sol. Equation of the plane through the lines $x + 2y - 4z = 0$ and $2x - y + 2z = 0$ is

$x + 2y - 4z + \lambda (2x - y + 2z) = 0$ ...(1)

If (1, 1, 1) lies on this plane, then $-1 + 3\lambda = 0$

$\Rightarrow \lambda = \frac{1}{3}$, so that the plane becomes $3x + 6y - 12z + 2x - y + 2z = 0 \Rightarrow x + y - 2z = 0$ ....(2)

Also (1) will be perpendicular to (2) if $1 + 2\lambda + 2 - \lambda - 2(-4 + 2\lambda) = \Rightarrow \lambda = \frac{11}{3}$.

$\Rightarrow$ Equation of plane perpendicular to (2) is $5x - y + 2z = 0$. ...(3)

Therefore the equation of line through (1, 1, 1) and perpendicular to the given line is parallel to the normal to the plane (3). Hence the required line is $\frac{x-1}{5} = \frac{y-1}{-1} = \frac{z-1}{2}$

Alternate :

Solving the equation of planes $x + 2y - 4z = 0$ and $2x - y + 2z = 0$, we get $\frac{x}{0} = \frac{y}{-10} = \frac{z}{-5}$ ...(1)

Any point P on the line (1) can be written as $(0, -10\lambda, -5\lambda)$.

Direction ratios of the line joining P and Q(1, 1, 1) is $(1, 1, + 10\lambda, 1 + 5\lambda)$.

Line PQ is perpendicular to line (1) $\Rightarrow$ $0(1) - 10(1 + 10\lambda) - 5(1 + 5\lambda) = 0$

$\Rightarrow 0 - 10 - 100\lambda - 5 - 25x = 0$ or $125\lambda + 15 = 0 \Rightarrow = \frac{-15}{125} = \frac{-3}{25}$ $\Rightarrow$ $P = \left(0, \frac{6}{5}, \frac{3}{5}\right)$

Direction ratios of PQ = $\left(-1, \frac{1}{5}, \frac{-2}{5}\right)$. Hence equations of lien are $\frac{x-1}{5} = \frac{y-1}{-1} = \frac{z-1}{2}$.

Ex.29 Find the shortest distance (S.D.) between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}, \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

Find also its equations and the points in which it meets the given lines.

Sol. The equations of the given lines are $(x - 3)/3 = (y - 8)/-1 = (z - 3)/1 = r_1$ (say) ...(1)

and $(x + 3)/(-3) = (y + 7)/2 = (z - 6)/4 = r_2$ (say) ...(2)

Any point on line (1) is $(3r_1 + 3, -r_1 + 8, r_1 + 3)$, say P. ...(3)

any point on line (2) is $(-3r_2 - 3, 2r_2 - 7, 4r_2 + 6)$, say Q. ...(4)

The d.r.'s of the line PQ are $(-3r_2 - 3) - (3r_1 + 3), (2r_2 - 7) - (-r_1 + 8), (4r_2 + 6) - (r_1 + 3)$

or $-3r_2 - 3r_1 - 6, 2r_2 + r_1 - 15, 4r_2 - r_1 + 3$. ...(5)

Let the line PQ be the lines of S.D., so that PQ is perpendicular to both the given lines (1) and (2), and so we have $3(-3r_2 - 3r_1 - 6) - 1(2r_2 + r_1 - 15) + 1.(4r_2 - r_1 + 3) = 0$

and $-3(-3r_2 - 3r_1 - 6) + 2.(2R_2 + r_1 - 15) + 4(4r_2 - r_1 + 3) = 0$

or $-7r_2 - 11r_1 = 0$ and $11r_2 + 7r_1 = 0$. Solving these equations, we get $r_1 = r_2 = 0$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have P(3, 8, 3), Q(–3, –7, 6)

And the d.r.'s of PQ (the line of S.D.) are –6, –15, 3 or –2, –5, 1.

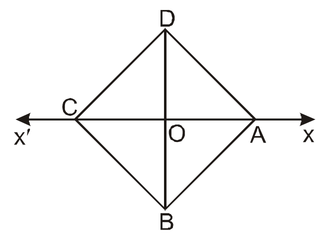
The length of S.D. = the distance between the points P and Q = $\sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2} = 3\sqrt{30}$.

Now the line PQ of shortest distance is the line passing through P(3, 8, 3) and having d.r.'s –2, –5, 1 and hence its equations are given by $\frac{x-3}{-2} = \frac{y-8}{-5} = \frac{z-3}{1}$ or $\frac{x-3}{2} - \frac{y-8}{5} = \frac{z-3}{1}$.

Ex.30 A square ABCD of diagonal 2a is folded along the diagonal AC so that the planes DAC, BAC are at right angles. Find the shortest distance between DC and AB.

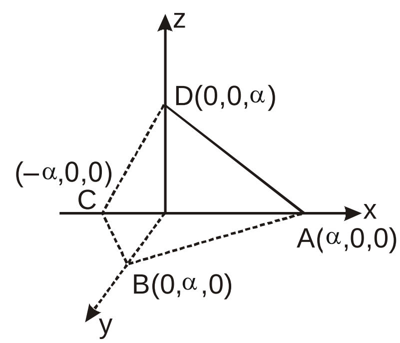
Sol. ABCD is a square of diagonal 2a, so that AC = BD = 2a. Let O, the centre of the square, be chosen as origin of co-ordinates and the diagonal CA be taken along x-axis. Hence the co-ordinates of the vertices A and C are

(a, 0, 0) and (–a, 0, 0) respectively.



Now as given in the problem, the square is folded over along the diagonal AC so that the planes DAC and BAC are at right angles. This implies that the lines OB and OD become at right angles. Also OA is perpendicular to the plane DOB. Hence the lines OA, OB, OD are mutually orthogonal. Let us now take OB and OD as y

and z axes respectively.



$\therefore$ The co-ordinates of B and D are (0, a, 0) and (0, 0, a) respectively.

The equations to AB are $\dfrac{x-a}{a} = \dfrac{y-0}{-a} = \dfrac{z-0}{0}$ .....(1)

The equation to DC are $\dfrac{x-0}{a} = \dfrac{y-0}{0} = \dfrac{z-a}{a}$ .....(2)

The equation of any plane through DC and parallel to AB [i.e. through the line (2) and parallel to the line (1)] is $\begin{vmatrix} x-0 & y-0 & z-0 \\ a & 0 & a \\ a & -a & 0 \end{vmatrix} = 0$ or $x(a^2) - y(-a^2) + (z - a)(-a^2) = 0$ or $x + y - z + a = 0$ ...(3)

$\therefore$ The S.D. between DC and AB

= the length of perpendicular from a point (a, 0, 0) on AB [i.e. (1)] to the plane (3)

$$= \frac{a+0-0+a}{\sqrt{\{(1)^2+(1)^2+(-1)^2\}}}\,\frac{2a}{\sqrt{3}}.$$

Ex.31 Find the condition that the equation $\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ may represent a pair of planes, passing through the origin

Sol. Since it passes through the origin, let it represent the planes

$\ell_1 x + m_1 y + n_1 z = 0$ ...(1) and $\ell_2 x + m_2 y + n_2 z = 0$ ...(2)

$\Rightarrow$ $a x^2 + b y^2 + c z^2\ 2 f y z + 2 g z x + 2 h x y \equiv (\ell_1 x + m_1 y + n_1 z)(\ell_2 x + m_2 y + n_2 z) = 0$

comparing the coefficients of x^2, y^2, z^2, yz, zx and xy of both sides, we get,

$\ell_1 \ell_2 = a$; $m_1 m_2 = b$; $n_1 n_2 = c$;

$m_1 n_2 + m_2 n_1 = 2f$; $n_2 \ell_2 + n_2 \ell_1 = 2g$ and $\ell_1 m_2 + \ell_2 m_1 = 2h$ ...(3)

consider the product of two zero determinants $\begin{vmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} = 0$ and $\begin{vmatrix} \ell_2 & \ell_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$

i.e. $\begin{vmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} \ell_2 & \ell_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$ or $\begin{vmatrix} 2\ell_1\ell_2 & \ell_1 m_2 + \ell_2 m_1 & \ell_1 n_2 + \ell_2 n_1 \\ \ell_1 m_2 + \ell_2 m_1 & 2m_1 m_2 & m_1 n_2 + m_2 n_1 \\ \ell_1 n_2 + \ell_2 n_1 & m_1 n_2 + m_2 n_1 & 2n_1 n_2 \end{vmatrix} = 0$

putting the values of $\ell_1\ \ell_2$, $m_1\ m_2$ ...... etc. from (4), we get

$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ i.e. $a\,b\,c + 2\,f\,g\,h - a\,f^2 - b\,g^2 - c\,h^2 = 0$

which is the required condition for ϕ (x, y, z) = 0 to represent pair of planes passing through origin.

Ex.32 Prove that the product of distances of the planes represented by

$\phi(x, y, z) = a\,x^2 + b\,y^2 + c\,z^2 + 2\,f\,y\,z + 2\,g\,z\,x + 2\,h\,x\,y = 0$ from (a, b, c) is $\left| \dfrac{\phi(a,b,c)}{\sqrt{\sum a^2 + 4\sum h^2 - 2\sum ab}} \right|$.

Sol. Let the equation of two planes be $\alpha_1 x + \beta_1 y + \gamma_1 z = 0$ and $\alpha_2 x + \beta_2 y + \gamma_2 z = 0$

So, that $\phi(x, y, z) \equiv (\alpha_1 x + \beta_1 y + \gamma_1 z)(\alpha_2 x + \beta_2 y + \gamma_2 z) = 0$ ....(1)

Comparing the coefficients, we get $\alpha_1\alpha_2 = a$, $\beta_1\beta_2 = b$, $\gamma_1\gamma_2 = c$

$\beta_1\gamma_2 + \beta_2\gamma_1 = 2f$; $\gamma_1\alpha_2 + \gamma_2\alpha_1 = 2g$; $\alpha_1\beta_2 + \beta_2\alpha_1 = 2h$

Let p_1 and p_2 be the perpendiculars distances of the point (a, b, c) from the two planes then

$$p_1\,p_2 = \left| \frac{\alpha_1 a + \beta_1 b + \gamma_1 c}{\sqrt{\alpha_1^2 + \beta_1^2 + \gamma_1^2}} \right| \left| \frac{\alpha_2 a + \beta_2 b + \gamma_2 c}{\sqrt{\alpha_2^2 + \beta_2^2 + \gamma_2^2}} \right|$$

$$= \left| \frac{(\alpha_1\alpha_2 a^2 + \beta_1\beta_2 b^2 + \gamma_1\gamma_2 c^2)(\alpha_1\beta_2 + \beta_1\alpha_2)ab + (\beta_1\gamma_2 + \beta_2\gamma_1)bc + (\alpha_1\gamma_2 + \alpha_2\gamma_1)ac}{\sqrt{\alpha_1^2\alpha_2^2 + \beta_1^2\beta_2^2 + \gamma_1^2\gamma_2^2 + (\alpha_1^2\beta_2^2 + \alpha_1^2\beta_2^2) + (\beta_1^2\gamma_2^2 + \gamma_1^2\beta_2^2) + (\gamma_1^2\alpha_2^2 + \alpha_1^2\gamma_2^2)}} \right|$$

$$= \left| \frac{a\,.\,a^2 + b\,.\,b^2 + c\,.\,c^2 + 2hab + 2fbc + 2gac}{\sqrt{a^2 + b^2 + c^2 + \sum[(\alpha_1\beta_2 + \beta_2\alpha_1)^2 - 2\alpha_1\alpha_2.\beta_1\beta_2]}} \right| = \left| \frac{\phi(a,b,c)}{\sqrt{\sum a^2 + \sum[4h^2 - 2ab]}} \right|$$

$$\Rightarrow \quad p_1 p_2 = \left| \frac{\phi(a,b,c)}{\sqrt{\sum a^2 + a\sum h^2 - 2\sum ab}} \right|.$$

Ex.33 From a point (1, 1, 21), a ball is dropped onto the plane x + y + z = 3, where x, y-plane is horizontal and z-axis is along the vertical. Find the co-ordinates of the point where the ball hits the plane the second time. (use $s = ut - 1/2gt^2$ and $g = 10\ m/s^2$)

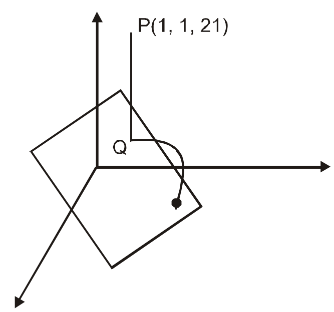
Sol. Since it falls along the vertical, the x-y coordinates of the ball will not change before it strikes the plane $\Rightarrow$ If Q be the point where the ball meets the plane 1st time, then $Q \equiv (1, 1, 1)$

Speed of the balls just before striking the plane is $\sqrt{2 \times 10 \times 20}$ = 20 m/s.

Now let θ be the angle between PQ and normal to the plane $\Rightarrow \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \cos 2\theta = -\frac{1}{3}$, $\sin 2\theta = \frac{2\sqrt{2}}{3}$

Now component of velocity in the direction of z-axis after it strikes the plane

$= -20 \sin\left(2\theta - \frac{\pi}{2}\right) = -\frac{20}{3}$ m/s



Hence in 't' time the z-coordinate of ball becomes

$1 - \frac{20}{3}t - \frac{1}{2} \times 10t^2 = 1 - \frac{20}{3}t - 5t^2$

The component of velocity in x-y plane is

$20 \cos\left(2\theta - \frac{\pi}{2}\right) = 20 \sin 2\theta = \frac{20 \times 2\sqrt{2}}{3} = \frac{40\sqrt{2}}{3}$

Using symmetry, the component along the x-axis = $\frac{40}{3}$ & the component along the y-axis = $\frac{40}{3}$

Hence x and y coordinates of the ball after t time = $1 + \frac{40}{3}t$

$\Rightarrow$ after t time the coordinate of the ball will become $\left(1 + \frac{40}{3}t, 1 + \frac{40}{3}t, 1 - \frac{20}{3}t - 5t^2\right)$

Its lies on the plane $\frac{80}{3}t - \frac{20}{3}t - 5t^2 = 0 \Rightarrow 20t - 5t^2 = 0 \Rightarrow t = 4$

$\Rightarrow$ coordinate of the point where the ball strikes the plane the second time = $\left[\frac{163}{3}, \frac{163}{3}, \frac{-317}{3}\right]$.